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Limit behavior of global attractors for the complex Ginzburg–Landau equation on infinite lattices☆

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Abstract

In this work, the authors first show the existence of global attractors $\mathcal{A}_\varepsilon$ for the following lattice complex Ginzburg–Landau equation:

$$\mathrm{i}\dot{u}_m - (\alpha - \mathrm{i}\varepsilon)(2u_m - u_{m+1} - u_{m-1}) + \mathrm{i}\kappa u_m + \beta|u_m|^{2\sigma} u_m = g_m, \quad m \in \mathbb{Z},\ \varepsilon > 0,$$

and $\mathcal{A}_0$ for the following lattice Schrödinger equation:

$$\mathrm{i}\dot{u}_m - \alpha(2u_m - u_{m+1} - u_{m-1}) + \mathrm{i}\kappa u_m + \beta|u_m|^{2\sigma} u_m = g_m, \quad m \in \mathbb{Z}.$$

Then they prove that the solutions of the lattice complex Ginzburg–Landau equation converge to that of the lattice Schrödinger equation as $\varepsilon \to 0+$. Also they prove the upper semicontinuity of $\mathcal{A}_\varepsilon$ as $\varepsilon \to 0+$ in the sense that $\lim_{\varepsilon \to 0+} \mathrm{dist}_{\ell^2}(\mathcal{A}_\varepsilon, \mathcal{A}_0) = 0$.

Keywords: Global attractor; Upper semicontinuity; Lattice systems; Complex Ginzburg–Landau equation; Schrödinger equation

1. Introduction

Recently, the dynamics of infinite lattice systems have drawn much attention from mathematicians and physicists; see [1–3,6,11,18,21] and the references therein. LDSs occur naturally in many applied sciences, such as electrical engineering [4], laser systems [10], image processing and pattern recognition [5,8,9], etc. In some cases, LDSs arise as the spatial discretization of partial differential equations (PDEs) on unbounded (or bounded) domains and they can be regarded as approximations to the corresponding continuous PDEs.

The theory of global attractors for PDEs is now a well-developed part of the modern theory of infinite dimensional dynamical systems (see [7,12,14–16]). There are many works concerning the global attractors for LDSs [1,2,11, 17,18,20,21]. For example, Bates et al. [1] presented a framework for the existence and upper semicontinuity of a

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global attractor associated with first-order LDSs. Later, Wang [18] and Zhou and Shi [21] obtained, respectively, some sufficient and necessary conditions for the existence of a global attractor for the semigroup corresponding to general LDSs. In particular, Vleck and Wang [17]discussed the singular limiting behavior of attractors for lattice FitzHugh–Nagumo systems.

In the present work, we consider the following lattice complex Ginzburg–Landau equation with initial values:

$$\mathrm{i}\dot{u}_m - (\alpha - \mathrm{i}\varepsilon)(2u_m - u_{m+1} - u_{m-1}) + \mathrm{i}\kappa u_m + \beta |u_m|^{2\sigma} u_m = g_m, \quad m \in \mathbb{Z}, \tag{1.1}$$

$$u_m(0) = u_{0,m}, \quad m \in \mathbb{Z}, \tag{1.2}$$

where $\alpha, \beta, \kappa, \sigma, \varepsilon$ are positive parameters and i is the unit of imaginary numbers such that $\mathrm{i}^2 = -1$. When $\varepsilon = 0$, (1.1) and (1.2) turn into the following lattice Schrödinger equation with initial values:

$$\mathrm{i}\dot{u}_m - \alpha(2u_m - u_{m+1} - u_{m-1}) + \mathrm{i}\kappa u_m + \beta |u_m|^{2\sigma} u_m = g_m, \quad m \in \mathbb{Z}, \tag{1.3}$$

$$u_m(0) = u_{0,m}, \quad m \in \mathbb{Z}. \tag{1.4}$$

We first show that Eqs. (1.1) and (1.2) have a unique solution $u = (u_m)_{m\in\mathbb{Z}} \in \mathcal{C}([0, +\infty); \ell^2)$, where

$$\ell^2 = \left\{ u = (u_m)_{m\in\mathbb{Z}} : u_m \in \mathbb{C}, \sum_{m\in\mathbb{Z}} |u_m|^2 < +\infty \right\}. \tag{1.5}$$

Then we prove that for each $\varepsilon > 0$ the solution operator semigroup of (1.1) and (1.2) possesses a unique global attractor $\mathcal{A}_\varepsilon \subset \ell^2$. Notice that [11] established that Eqs. (1.3) and (1.4) possess a unique global attractor $\mathcal{A}_0 \subset \ell^2$; there are two basic questions as regards (1.1), (1.2) and (1.3), (1.4): Do the solutions of (1.1) and (1.2) tend to the solutions of (1.3) and (1.4) as the parameter $\varepsilon \to 0+$? Does the global attractor $\mathcal{A}_\varepsilon$ approximate the global attractor $\mathcal{A}_0$ as $\varepsilon \to 0+$?

We give positive answers to the above questions within this work. Firstly, we verify the continuous dependence of solutions of (1.1) and (1.2) as $\varepsilon \to 0+$. Secondly, we use the continuous dependence to prove the upper semicontinuity of the global attractor $\mathcal{A}_\varepsilon$ in the sense that $\lim_{\varepsilon\to 0+} \mathrm{dist}_{\ell^2}(\mathcal{A}_\varepsilon, \mathcal{A}_0) = 0$, where $\mathrm{dist}_{\ell^2}(X, Y) = \sup_{x\in X} \inf_{y\in Y} \|x - y\|$ denotes the Hausdorff semidistance in ℓ^2.

We remark that the idea of this work originates from papers [17,19]. In [19] Wang studied systematically the limit behavior of solutions for the Cauchy problem of the following continuous complex Ginzburg–Landau equation:

$$u_t - \varepsilon \Delta u - \mathrm{i}\Delta u + (a + \mathrm{i})|u|^\alpha u = 0, \qquad u(0, x) = u_0(x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n. \tag{1.6}$$

In fact, Eq. (1.1) can be regarded as a discrete analogue to the following weakly damped complex Ginzburg–Landau equation on $\mathbb{R}$:

$$\mathrm{i}u_t + (\alpha - \mathrm{i}\varepsilon)\Delta u + \mathrm{i}\kappa u + \beta |u|^{2\sigma} u = g. \tag{1.7}$$

There are many works concerning Eq. (1.6) or (1.7); see [13,19] and the references therein.

We also would like to point out that, although LDSs (in some cases) can be regarded as approximations to the corresponding continuous PDEs, there is an intrinsic difference between them. In general, the spatial dimension plays an essential role in the study of PDEs; this is mainly caused by the fact that many important tools (such as Sobolev space, Strichartz-type inequality) used in studying PDEs depend on the spatial dimension; see for example [19]. But when one considers LDSs, spatial dimension does not play a crucial role; see Remark 4.2 later.

The work is organized as follows. In the next section, we show the existence, uniqueness and boundedness of solutions to Eqs. (1.1)–(1.4). In Section 3, we prove that for each $\varepsilon > 0$ Eqs. (1.1) and (1.2) possess a unique global attractor $\mathcal{A}_\varepsilon \subset \ell^2$. In the last section, we establish the continuous dependence of solutions of (1.1) and (1.2) and the upper semicontinuity of the global attractor $\mathcal{A}_\varepsilon$ as $\varepsilon \to 0+$.

2. Uniqueness, existence and boundedness

For any $u = (u_m)_{m\in\mathbb{Z}}, v = (v_m)_{m\in\mathbb{Z}} \in \ell^2$, we define

$$(u, v) = \sum_{m\in\mathbb{Z}} u_m \bar{v}_m, \qquad \|u\|^2 = (u, u),$$

where $\bar{v}_m$ denotes the conjugate number of v_m. Then $\ell^2 = (\ell^2, (\cdot,\cdot), \|\cdot\|)$ is a Hilbert space. Define some linear operators from ℓ^2 to ℓ^2 as follows: for any $u = (u_m)_{m\in\mathbb{Z}} \in \ell^2$,

$$(Au)_m = 2u_m - u_{m+1} - u_{m-1}, \qquad (Bu)_m = u_{m+1} - u_m, \qquad (B^*u)_m = u_{m-1} - u_m.$$

Then B^* is the adjoint operator of B, and

$$A = B^*B = BB^*. \tag{2.1}$$

Moreover, we have

$$\|Au\|^2 \le 16\|u\|^2, \qquad \|Bu\|^2 = \|B^*u\|^2 \le 4\|u\|^2, \quad \forall u \in \ell^2. \tag{2.2}$$

For convenience, we express Eqs. (1.1) and (1.2) as an initial value problem in the Hilbert space ℓ^2:

$$\mathrm{i}\dot{u} - (\alpha - \mathrm{i}\varepsilon)Au + \mathrm{i}\kappa u + \beta|u|^{2\sigma}u = g, \quad t > 0, \tag{2.3}$$

$$u(0) = u_0, \tag{2.4}$$

where $u = (u_m)_{m\in\mathbb{Z}}$, $|u|^{2\sigma}u = (|u_m|^{2\sigma}u_m)_{m\in\mathbb{Z}}$, $g = (g_m)_{m\in\mathbb{Z}}$, $u_0 = (u_{0,m})_{m\in\mathbb{Z}}$.

Lemma 2.1. *Let $\alpha, \beta, \kappa, \varepsilon, \sigma$ be positive and $g = (g_m)_{m\in\mathbb{Z}} \in \ell^2$. Then for any $u_0 \in \ell^2$, Eqs. (2.3) and (2.4) admit a unique solution $u \in \mathcal{C}([0, T_0), \ell^2)$ for some $T_0 > 0$. Moreover, if $T_0 < +\infty$, then $\lim_{t\to T_0-} \|u(t)\| = +\infty$.*

Proof. We rewrite Eq. (2.3) as

$$\dot{u} = F(u) \triangleq (-\varepsilon - \mathrm{i}\alpha)Au - \kappa u + \mathrm{i}\beta f(|u|^2)u - \mathrm{i}g, \tag{2.5}$$

where

$$f(x) = x^\sigma, \quad x \in \mathbb{R}_+. \tag{2.6}$$

Then we need to check that $F(\cdot): \ell^2 \to \ell^2$ is Lipschitz continuous on bounded sets of ℓ^2. Since $f'(x) = \sigma x^{\sigma-1}$ is continuous on $\mathbb{R}_+$, by Lemma 2.1 in [11], we see that $\tilde{f}(u) = f(|u|^2)u$ is Lipschitz continuous on bounded sets of ℓ^2. Thus $F(u)$ is Lipschitz continuous on bounded sets of ℓ^2 because A is a bounded linear operator on ℓ^2. We then obtain Lemma 2.1 by the classical theory of ODEs. The proof is complete. □

Lemma 2.2. *Let $\alpha, \beta, \kappa, \varepsilon, \sigma$ be positive and $g = (g_m)_{m\in\mathbb{Z}} \in \ell^2$. Then for any $u_0 \in \ell^2$, the solution of Eqs. (2.3) and (2.4) satisfies*

$$\|u(t)\|^2 \le \|u_0\|^2 \mathrm{e}^{-\kappa t} + \frac{\|g\|^2}{\kappa^2}, \quad \forall t \ge 0. \tag{2.7}$$

Proof. Taking the real part of the inner product $(\cdot,\cdot)$ of (2.3) with $\mathrm{i}u(t)$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \varepsilon\|Bu\|^2 + \kappa\|u\|^2 = \mathbf{Im}\sum_{m\in\mathbb{Z}} g_m\bar{u}_m \le \frac{\kappa}{2}\|u\|^2 + \frac{1}{2\kappa}\|g\|^2,$$

from which we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \kappa\|u\|^2 \le \frac{1}{\kappa}\|g\|^2. \tag{2.8}$$

Applying the Gronwall inequality to (2.8), we obtain (2.7) and end the proof. □

Letting $t \to +\infty$ in (2.7), we see that for any $u_0 \in \ell^2$, the corresponding solution $u(t) \in \ell^2$ of Eqs. (2.3) and (2.4) is uniform (with respect to ε) bounded for all $t \in [0, +\infty)$, which implies that for each $\varepsilon > 0$, the solution operators

$$S_\varepsilon(t) : u_0 \in \ell^2 \mapsto u(t) = S_\varepsilon(t)u_0 \in \ell^2, \quad t \ge 0, \tag{2.9}$$

generate a continuous semigroup $\{S_\varepsilon(t)\}_{t\ge0}$ on ℓ^2. Obviously, by Lemmas 2.1 and 2.2, we have

Lemma 2.3. *Let α, β, κ, ε, σ be positive and $g = (g_m)_{m\in\mathbb{Z}} \in \ell^2$. Then for each $\varepsilon > 0$, the semigroup $\{S_\varepsilon(t)\}_{t\geq 0}$ corresponding to Eqs. (2.3) and (2.4) possesses a uniform (with respect to ε) bounded absorbing set $\mathcal{B}_0$: for any bounded set $\mathcal{B} \subset \ell^2$, there exists a time $t(\mathcal{B}) > 0$ such that*

$$S_\varepsilon(t)\mathcal{B} \subset \mathcal{B}_0, \quad \forall t \geq t(\mathcal{B}),\ \forall \varepsilon > 0,$$

where $\mathcal{B}_0 = \mathcal{B}(0, R)$ is a closed ball centered at zero with radius $R = \frac{\sqrt{2}\|g\|}{\kappa}$, which is independent of ε.

From Lemma 2.3 we see that there exists a time $t_0 = t_0(\mathcal{B}_0) > 0$ such that

$$S_\varepsilon(t)\mathcal{B}_0 \subset \mathcal{B}_0, \quad \forall t \geq t_0,\ \forall \varepsilon > 0. \tag{2.10}$$

3. Existence of a global attractor

In this section, we verify the existence of a global attractor for the semigroup $\{S_\varepsilon(t)\}_{t\geq 0}$ defined by (2.9). To this end, we establish that $\{S_\varepsilon(t)\}_{t\geq 0}$ has asymptotical end tail.

Definition 3.1 (*[21]*). $\{S_\varepsilon(t)\}_{t\geq 0}$ is said to have asymptotical end tail in ℓ^2 if for any $\eta > 0$, there exist $T(\eta) > 0$ and $M(\eta) \in \mathbb{N}$ (maybe depending on $\mathcal{B}_0$) such that the solution $u(t)$ of Eqs. (2.3) and (2.4) with initial value $u_0 \in \mathcal{B}_0$ satisfies

$$\sum_{|m|>M(\eta)} |u_m(t)|^2 \leq \eta, \quad \forall t \geq T(\eta).$$

Lemma 3.1. *Let α, β, κ, ε, σ be positive and $g = (g_m)_{m\in\mathbb{Z}} \in \ell^2$. Then for each $\varepsilon > 0$, the semigroup $\{S_\varepsilon(t)\}_{t\geq 0}$ has asymptotical end tail in ℓ^2.*

Proof. Define a smooth function $\chi(x) \in \mathcal{C}(\mathbb{R}_+, [0, 1]) \cap \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$ such that

$$\chi(x) = \begin{cases} 0, & 0 \leq x \leq 1; \\ 1, & x \geq 2, \end{cases} \quad \text{and} \quad \chi'(x) \leq \chi_0 \text{ (positive constant)}, \quad \forall x \in \mathbb{R}_+. \tag{3.1}$$

Let M be some positive integer and set $\xi = (\xi_m)_{m\in\mathbb{Z}}$ with $\xi_m = \chi(\frac{|m|}{M})u_m$, $m \in \mathbb{Z}$. Taking the real part of the inner product $(\cdot,\cdot)$ of (2.3) with $\mathrm{i}\xi(t)$, we obtain

$$\mathbf{Re}\sum_{m\in\mathbb{Z}} \dot{u}_m\bar{\xi}_m + \mathbf{Re}(\mathrm{i}\alpha+\varepsilon)\sum_{m\in\mathbb{Z}}(Bu)_m(B\bar{\xi})_m + \kappa\sum_{m\in\mathbb{Z}} u_m\bar{\xi}_m = \mathbf{Im}\sum_{m\in\mathbb{Z}} g_m\bar{\xi}_m. \tag{3.2}$$

We next deal with the four terms in (3.2) one by one. Firstly,

$$\mathbf{Re}\sum_{m\in\mathbb{Z}} \dot{u}_m\bar{\xi}_m = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)|u_m|^2. \tag{3.3}$$

Secondly, since

$$\begin{aligned} &\varepsilon\sum_{m\in\mathbb{Z}}\left(\chi\left(\frac{|m+1|}{M}\right)|u_{m+1}|^2 + \chi\left(\frac{|m|}{M}\right)|u_m|^2\right) \\ &\quad \geq \mathbf{Re}\,\varepsilon\sum_{m\in\mathbb{Z}}\left(\chi\left(\frac{|m+1|}{M}\right)\bar{u}_{m+1}u_m + \chi\left(\frac{|m|}{M}\right)\bar{u}_m u_{m+1}\right), \end{aligned} \tag{3.4}$$

we have

$$\begin{aligned} \mathbf{Re}(\mathrm{i}\alpha+\varepsilon)\sum_{m\in\mathbb{Z}}(Bu)_m(B\bar{\xi})_m &= \mathbf{Re}(\mathrm{i}\alpha+\varepsilon)\sum_{m\in\mathbb{Z}}(u_{m+1}-u_m)\left(\chi\left(\frac{|m+1|}{M}\right)\bar{u}_{m+1} - \chi\left(\frac{|m|}{M}\right)\bar{u}_m\right) \\ &= \varepsilon\sum_{m\in\mathbb{Z}}\left(\chi\left(\frac{|m+1|}{M}\right)|u_{m+1}|^2 + \chi\left(\frac{|m|}{M}\right)|u_m|^2\right) \end{aligned}$$

$$
\begin{aligned}
&+\mathbf{Im}\,\alpha\sum_{m\in\mathbb{Z}}\left(\chi\left(\frac{|m+1|}{M}\right)\bar{u}_{m+1}u_m+\chi\left(\frac{|m|}{M}\right)u_{m+1}\bar{u}_m\right)\\
&-\mathbf{Re}\,\varepsilon\sum_{m\in\mathbb{Z}}\left(\chi\left(\frac{|m+1|}{M}\right)\bar{u}_{m+1}u_m+\chi\left(\frac{|m|}{M}\right)u_{m+1}\bar{u}_m\right)\quad\text{(by (3.4))}\\
\geq\ &\mathbf{Im}\,\alpha\sum_{m\in\mathbb{Z}}\left(\chi\left(\frac{|m+1|}{M}\right)\bar{u}_{m+1}u_m+\chi\left(\frac{|m|}{M}\right)u_{m+1}\bar{u}_m\right)\\
=\ &\mathbf{Im}\,\alpha\sum_{m\in\mathbb{Z}}\left(\chi\left(\frac{|m+1|}{M}\right)-\chi\left(\frac{|m|}{M}\right)\right)\bar{u}_{m+1}u_m\\
=\ &\mathbf{Im}\,\alpha\sum_{m\in\mathbb{Z}}\chi'\left(\frac{\tilde{m}}{M}\right)\frac{1}{M}\bar{u}_{m+1}u_m\\
\geq\ &-\frac{\alpha\chi_0}{M}\|u\|^2\quad\text{(by (2.10))}\\
\geq\ &-\frac{\alpha\chi_0R^2}{M},\quad\forall t\geq t_0,
\end{aligned}\tag{3.5}
$$

and hereafter $\tilde{m}$ denotes some constant locating between $|m+1|$ and $|m|$, R and t_0 come from Lemma 2.3 and (2.10), respectively. Lastly,

$$\kappa\sum_{m\in\mathbb{Z}}u_m\bar{\xi}_m=\kappa\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)|u_m|^2,\tag{3.6}$$

$$\mathbf{Im}\sum_{m\in\mathbb{Z}}g_m\bar{\xi}_m\leq\frac{\kappa}{2}\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)|u_m|^2+\frac{1}{2\kappa}\sum_{|m|\geq M}|g_m|^2.\tag{3.7}$$

Combining (3.2), (3.3) and (3.5)–(3.7), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)|u_m|^2+\kappa\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)|u_m|^2\leq\frac{1}{\kappa}\sum_{|m|\geq M}|g_m|^2+\frac{2\alpha\chi_0R^2}{M},\quad\forall t\geq t_0.\tag{3.8}$$

Applying the Gronwall inequality to (3.8), we have for any $t\geq t_0$ that

$$\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)|u_m(t)|^2\leq\mathrm{e}^{-\kappa(t-t_0)}\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)|u_m(t_0)|^2+\frac{1}{\kappa^2}\sum_{|m|\geq M}|g_m|^2+\frac{2\alpha\chi_0R^2}{M\kappa}.\tag{3.9}$$

Now for any $\eta>0$, since $g=(g_m)_{m\in\mathbb{Z}}\in\ell^2$, there exists $M(\eta)\in\mathbb{N}$ such that

$$\frac{1}{\kappa}\sum_{|m|\geq M}|g_m|^2+\frac{2\alpha\chi_0R^2}{M}\leq\frac{\kappa\eta}{2},\quad\forall M\geq M(\eta).\tag{3.10}$$

Also, by (2.10) there exists $T(\eta)=t_0+\frac{1}{\kappa}\ln(\frac{2R^2}{\kappa\eta})$ such that

$$\mathrm{e}^{-\kappa(t-t_0)}\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)|u_m(t_0)|^2\leq R^2\mathrm{e}^{-\kappa(t-t_0)}\leq\frac{\eta}{2},\quad\forall t\geq T(\eta).\tag{3.11}$$

It then follows from (3.9)–(3.11) that

$$\sum_{|m|\geq 2M}|u_m(t)|^2\leq\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)|u_m(t)|^2\leq\eta,\quad\forall M\geq M(\eta),\ \forall t\geq T(\eta).$$

The proof is complete. □

From Theorem 1 in [21] and Lemmas 2.3, 3.1, we conclude the following result of this section.

Theorem 3.1. *Let α, β, κ, ε, σ be positive and $g = (g_m)_{m\in\mathbb{Z}} \in \ell^2$. Then for each $\varepsilon > 0$, the semigroup $\{S_\varepsilon(t)\}_{t\ge 0}$ corresponding to Eqs. (1.1) and (1.2) possesses a unique global attractor $\mathcal{A}_\varepsilon \subset \mathcal{B}_0 \subset \ell^2$.*

4. Limit behavior of global attractors

In this section, we prove the main result of this work, which is the continuous dependence of solutions of Eqs. (1.1) and (1.2) and the upper semicontinuity of the global attractor $\mathcal{A}_\varepsilon$ as $\varepsilon \to 0+$.

We first recall some results on the limiting equations (for $\varepsilon = 0$) (1.3) and (1.4).

Lemma 4.1. *Let α, β, κ, ε, σ be positive and $g = (g_m)_{m\in\mathbb{Z}} \in \ell^2$. Then for any $u_0 \in \ell^2$, Eqs. (1.3) and (1.4) admit a unique solution $u(t) \in \mathcal{C}([0, +\infty), \ell^2)$ and the solution operators $S(t) : u_0 \mapsto u(t) = S(t)u_0$ form a continuous semigroup $\{S(t)\}_{t\ge 0}$ on ℓ^2. Moreover, $\{S(t)\}_{t\ge 0}$ possesses a unique global attractor $\mathcal{A}_0 \subset \ell^2$.*

The following Theorem 4.1 answers the first question formulated in the introduction.

Theorem 4.1. *Assume that α, β, κ, ε, σ are positive and $g = (g_m)_{m\in\mathbb{Z}} \in \ell^2$. Then for any $\varepsilon > 0$ and any $T > 0$, there exists a positive constant $C = C(\sigma, \kappa, \beta, R, T, \|g\|)$ such that*

$$\|S_\varepsilon(t)u_0 - S(t)u_0\| \le C\varepsilon, \quad \text{for any } t \in [0, T] \text{ and } u_0 \in \mathcal{B}_0. \tag{4.1}$$

Proof. Let $u_0 \in \mathcal{B}_0$ (the absorbing set given in Lemma 2.3). Set $u^\varepsilon = S_\varepsilon(t)u_0$ and $u(t) = S(t)u_0$. Then $v(t) = u^\varepsilon(t) - u(t) = S_\varepsilon(t)u_0 - S(t)u_0$ is a solution of the following problem:

$$\mathrm{i}\dot{v} - \alpha A v + \mathrm{i}\varepsilon A u^\varepsilon + \mathrm{i}\kappa v + \beta|u^\varepsilon|^{2\sigma}u^\varepsilon - \beta|u|^{2\sigma}u = 0, \tag{4.2}$$

$$v(0) = 0. \tag{4.3}$$

Taking the inner product $(\cdot,\cdot)$ of (4.2) with $\mathrm{i}v$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 + \mathbf{Re}\,\varepsilon(Bu^\varepsilon, Bv) + \kappa\|v\|^2 + \mathbf{Im}\,\beta\sum_{m\in\mathbb{Z}}(|u_m^\varepsilon|^{2\sigma}u_m^\varepsilon - |u_m|^{2\sigma}u_m)\bar{v}_m = 0. \tag{4.4}$$

Now by (2.2) and (2.7), we have

$$\left|\mathbf{Re}\,\varepsilon(Bu^\varepsilon, Bv)\right| \le 4\varepsilon\|u^\varepsilon\|\,\|v\| \le \frac{\kappa}{4}\|v\|^2 + \frac{32\varepsilon^2}{\kappa}\|u^\varepsilon\|^2 \le \frac{\kappa}{4}\|v\|^2 + \frac{32}{\kappa}\left(\|u_0\|^2 + \frac{\|g\|^2}{\kappa^2}\right)\varepsilon^2. \tag{4.5}$$

We next treat the term $\mathbf{Im}\,\beta\sum_{m\in\mathbb{Z}}(|u_m^\varepsilon|^{2\sigma}u_m^\varepsilon - |u_m|^{2\sigma}u_m)\bar{v}_m$. In fact,

$$\left|\mathbf{Im}\,\beta\sum_{m\in\mathbb{Z}}(|u_m^\varepsilon|^{2\sigma}u_m^\varepsilon - |u_m|^{2\sigma}u_m)\bar{v}_m\right| \le \frac{\kappa}{4}\|v\|^2 + \frac{2\beta^2}{\kappa}\sum_{m\in\mathbb{Z}}(|u_m^\varepsilon|^{2\sigma}u_m^\varepsilon - |u_m|^{2\sigma}u_m)^2, \tag{4.6}$$

and by (2.6) and (2.7), we have

$$\begin{aligned}
\sum_{m\in\mathbb{Z}}(|u_m^\varepsilon|^{2\sigma}u_m^\varepsilon - |u_m|^{2\sigma}u_m)^2 &= \sum_{m\in\mathbb{Z}}(f(|u_m^\varepsilon|^2)u_m^\varepsilon - f(|u_m|^2)u_m)^2 \\
&\le 2\sum_{m\in\mathbb{Z}}|f(|u_m^\varepsilon|^2)|^2|u_m^\varepsilon - u_m|^2 + 2\sum_{m\in\mathbb{Z}}|f(|u_m^\varepsilon|^2) - f(|u_m|^2)|^2|u_m|^2 \\
&\le 2\left(\max_{x\in[0, R^2+\frac{\|g\|^2}{\kappa^2}]} f(x)\right)^2 \sum_{m\in\mathbb{Z}}|u_m^\varepsilon - u_m|^2 + 2\sum_{m\in\mathbb{Z}}|f(|u_m^\varepsilon|^2) - f(|u_m|^2)|^2|u_m|^2 \\
&\le 2\left(R^2 + \frac{\|g\|^2}{\kappa^2}\right)^{2\sigma}\|u^\varepsilon - u\|^2 + 2\left(R^2 + \frac{\|g\|^2}{\kappa^2}\right)\underline{\sum_{m\in\mathbb{Z}}|f(|u_m^\varepsilon|^2) - f(|u_m|^2)|^2}.
\end{aligned} \tag{4.7}$$

We need to estimate the underlined term in (4.7). By the Mean Value Theorem, we have

$$\begin{aligned}\sum_{m\in\mathbb{Z}}|f(|u_m^\varepsilon|^2)-f(|u_m|^2)|^2 &= \sum_{m\in\mathbb{Z}}|f'(\theta|u_m^\varepsilon|^2+(1-\theta)|u_m|^2)|(|u_m^\varepsilon|+|u_m|)^2(|u_m^\varepsilon|-|u_m|)^2\\ &\le \left(\max_{x\in[0,R^2+\frac{\|g\|^2}{\kappa^2}]} f'(x)\right)^2 2\sum_{m\in\mathbb{Z}}(|u_m^\varepsilon|^2+|u_m|^2)(|u_m^\varepsilon|-|u_m|)^2\\ &\le 4\sigma^2\left(R^2+\frac{\|g\|^2}{\kappa^2}\right)^{2\sigma-1}\sum_{m\in\mathbb{Z}}|u_m^\varepsilon-u_m|^2\\ &= 4\sigma^2\left(R^2+\frac{\|g\|^2}{\kappa^2}\right)^{2\sigma-1}\|v\|^2. \end{aligned}\tag{4.8}$$

Combining (4.4)–(4.8), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 \le C_1\|v\|^2 + C_2\varepsilon^2,\tag{4.9}$$

where $C_1 = 8(1+4\sigma^2)\frac{\beta^2(R^2+\frac{\|g\|^2}{\kappa^2})^{2\sigma}}{\kappa} - \kappa$ and $C_2 = \frac{64}{\kappa}(R^2+\frac{\|g\|^2}{\kappa^2})$ are independent of ε. Applying the Gronwall inequality to (4.9), we have

$$\begin{aligned}\|v\|^2 = \|S_\varepsilon(t)u_0 - S(t)u_0\|^2 &\le \|v_0\|^2\mathrm{e}^{C_1 t} + C_2\varepsilon^2\int_0^t \mathrm{e}^{C_1(t-s)}\mathrm{d}s = C_2\varepsilon^2\int_0^t \mathrm{e}^{C_1(t-s)}\mathrm{d}s\\ &\le \frac{C_2}{C_1}\mathrm{e}^{C_1 t}\varepsilon^2 \le \frac{C_2}{C_1}\mathrm{e}^{C_1 T}\varepsilon^2, \quad \forall t\in[0,T].\end{aligned}$$

The proof is complete. □

Remark 4.1. If $C_1 < 0$, we have $\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 + (-C_1)\|v\|^2 \le C_2\varepsilon^2$, from which one can easily obtain (4.1).

Next we use Theorem 4.1 to prove the upper semicontinuity of the global attractor $\mathcal{A}_\varepsilon$ as $\varepsilon \to 0+$.

Theorem 4.2. *Assume that α, β, κ, ε, σ are positive and $g = (g_m)_{m\in\mathbb{Z}} \in \ell^2$. Then the global attractor $\mathcal{A}_\varepsilon$ of $\{S_\varepsilon(t)\}_{t\ge 0}$ corresponding to Eqs. (1.1) and (1.2) is upper semicontinuous at $\varepsilon = 0$ in the following sense:*

$$\lim_{\varepsilon\to 0+}\mathrm{dist}_{\ell^2}(\mathcal{A}_\varepsilon, \mathcal{A}_0) = 0.\tag{4.10}$$

Proof. On the one hand, for any $\varepsilon > 0$, $\mathcal{A}_\varepsilon \subset \mathcal{B}_0$ is uniform (with respect to ε) bounded in ℓ^2. On the other hand, $\mathcal{A}_0$ attracts any bounded set of ℓ^2. Thus for any $\eta > 0$, there exists some $T(\eta) > 0$ such that

$$\mathrm{dist}_{\ell^2}(S(T(\eta))\mathcal{A}_\varepsilon, \mathcal{A}_0) < \frac{\eta}{2}, \quad \forall \varepsilon > 0.\tag{4.11}$$

At the same time, for above $T(\eta) > 0$, Theorem 4.1 shows that there exists some $C = C(\sigma, \kappa, \beta, R, T(\eta), \|g\|) > 0$ such that

$$\mathrm{dist}_{\ell^2}(\mathcal{A}_\varepsilon, S(T(\eta))\mathcal{A}_\varepsilon) = \mathrm{dist}_{\ell^2}(S_\varepsilon(T(\eta))\mathcal{A}_\varepsilon, S(T(\eta))\mathcal{A}_\varepsilon) < C\varepsilon, \quad \forall\varepsilon > 0,\tag{4.12}$$

where we use that the invariance property of the global attractor $\mathcal{A}_\varepsilon : S_\varepsilon(t)\mathcal{A}_\varepsilon = \mathcal{A}_\varepsilon$ holds for any $t \ge 0$. It then follows from (4.11) and (4.12) that

$$\begin{aligned}\mathrm{dist}_{\ell^2}(\mathcal{A}_\varepsilon, \mathcal{A}_0) &\le \mathrm{dist}_{\ell^2}(\mathcal{A}_\varepsilon, S(T(\eta))\mathcal{A}_\varepsilon) + \mathrm{dist}_{\ell^2}(S(T(\eta))\mathcal{A}_\varepsilon, \mathcal{A}_0)\\ &< \mathrm{dist}_{\ell^2}(S_\varepsilon(T(\eta))\mathcal{A}_\varepsilon, S(T(\eta))\mathcal{A}_\varepsilon) + \frac{\eta}{2}\\ &< C\varepsilon + \frac{\eta}{2}, \quad \forall\eta > 0.\end{aligned}$$

Choose $\varepsilon_0 = \frac{\eta}{2C}$ and we get

$$\mathrm{dist}_{\ell^2}(\mathcal{A}_\varepsilon, \mathcal{A}_0) < \eta, \quad \forall \varepsilon \in [0, \varepsilon_0].$$

By the arbitrariness of η, we obtain (4.10) and end the proof of Theorem 4.2. □

Remark 4.2. When ℓ^2 is defined by

$$\ell^2 = \left\{ u = (u_m)_{m\in\mathbb{Z}^k} : m = (m_1, m_2, \ldots, m_k) \in \mathbb{Z}^k, u_m \in \mathbb{C}, \sum_{m\in\mathbb{Z}^k} |u_m|^2 < +\infty \right\},$$

the result of this work can be extended in a similar way. Here we will not pursue the details; one can refer to [20].

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